

# Localization of Nonlocal Cosmological Models with Quadratic Potentials in the case of Double Roots

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## Abstract

Nonlocal cosmological models with quadratic potentials are considered. We study the action with an arbitrary analytic function  $\mathcal{F}(\square_g)$ , which has both double and simple roots. The formulae for nonlocal energy–momentum tensor, which correspond to double roots, have been obtained. The way to find particular solutions for nonlocal Einstein equations in the case when  $\mathcal{F}(\square_g)$  has both simple and double roots has been proposed. One and the same functions solve the initial nonlocal Einstein equations and the obtained local Einstein equations.

## 1 Introduction

Recently a wide class of nonlocal cosmological models based on the string field theory (SFT) (for details see reviews [1]) and the  $p$ -adic string theory [2] emerges and attracts a lot of attention [3]–[18]. The SFT inspired cosmological models are intensively considered as models for dark energy (DE). Actions of some of cosmological models originating from the SFT have terms with infinite order derivatives, i.e. nonlocal terms.

Due to the presence of phantom excitations nonlocal models are of interest for the present cosmology. The inequality  $w_{\text{DE}} < -1$ , where  $w_{\text{DE}}$  is the DE state parameter, means the violation of the null energy condition (NEC). Field theories which violate the null energy condition are actively studied as a possible solution of the cosmological singularity problem [19, 20, 21] and as models of dark energy (see [22]–[31] and references therein). Generally speaking, models that violate the NEC have ghosts, and therefore are unstable and physically unacceptable. Phantom fields look harmful to the theory and a local model with a phantom scalar field is not acceptable from the general point of view. Models with higher derivative terms produce well-known problems with quantum instability [32, 33]. Several attempts

to solve these problems have been recently performed [34, 35]. A physical idea that could solve the problems is that the instabilities do not have enough time to fully develop. A mathematical one is that dangerous terms can be treated as corrections valued only at small energies below the physical cut-off. This approach implies the possibility to construct a UV completion of the theory, and this assumption requires detailed analysis.

Note that the possibility of the existence of dark energy with  $w_{\text{DE}} < -1$  is not excluded experimentally. Indeed, contemporary cosmological observational data [36] strongly support that the present Universe exhibits an accelerated expansion providing thereby an evidence for a dominating DE component (for a review see also [37]). Recent results of WMAP together with the data on Ia supernovae give the following bounds for the DE state parameter

$$w_{\text{DE}} = -1.0 \pm 0.2. \quad (1)$$

The present cosmological observations do not exclude an evolving parameter  $w_{\text{DE}}$ . Moreover, the recent analysis of the observation data indicates that the varying in time dark energy with the state parameter  $w_{\text{DE}}$ , which crosses the cosmological constant barrier, gives a better fit than a cosmological constant [38] (for details see reviews [39] and references therein).

To obtain a stable model with  $w_{\text{DE}} < -1$  one should construct the effective theory with the NEC violation from the fundamental theory, which is stable and admits quantization. From this point of view the NEC violation might be a property of a model that approximates the fundamental theory and describes some particular features of the fundamental theory. With the lack of quantum gravity, we can just trust string theory or deal with an effective theory admitting the UV completion. It can be considered as a hint towards the SFT inspired cosmological models.

Among cosmological models with  $w_{\text{DE}} < -1$ , which have been constructed to be free of instability problem, we can mention the Lorentz-violating dark energy model [40], the  $k$ -essence models (see [35] and references therein) and the brane-world models [41].

For a more general discussion on the string cosmology and coming out of string theory theoretical explanations of the cosmological observational data the reader is referred to [42, 43, 44]. Other models obeying nonlocality and their cosmological consequences are considered in [45]. In the flat space-time nonlocal equations are actively investigated as well [46, 47, 48, 49]. Note that differential equations of infinite order were begin to study in the mathematical literature long time ago [50, 51] (see [13] as a review).

The purpose of this paper is to study the string field theory inspired nonlocal model with a quadratic potential. In this paper we consider a general form of linear nonlocal action for the scalar field keeping the main ingredient, the function  $\mathcal{F}(\Box_g)$ , which in fact produces the nonlocality in question, almost unrestricted. The only strong restriction we impose is the analyticity of  $\mathcal{F}(\Box_g)$ . In previous papers [7, 8, 14, 15, 17] only simple roots have been considered. In this paper we consider the case of the function  $\mathcal{F}(\Box_g)$  with both simple and double roots.

The possible way to find solutions of the Einstein equations with a quadratic potential of the nonlocal scalar field, is to reduce them to a system of Einstein equations describing many non-interacting free local scalar fields [7, 14] (see also [17]). The masses of all local fields are roots of an algebraic or transcendental equation, which appears in the nonlocal

model. Some of the obtained local scalar fields are normal and other of them are phantom ones.

The particular forms of  $\mathcal{F}(\square_g)$  are inspired by the fermionic SFT and the most well understood process of tachyon condensation. Namely, starting with a non-supersymmetric configuration the tachyon of the fermionic string rolls down towards the nonperturbative minimum of the tachyon potential. This process represents the non-BPS brane decay according to Sen's conjecture (see [1] for details). From the point of view of the SFT the whole picture is not yet known and only vacuum solutions were constructed. An effective field theory description explaining the rolling tachyon in contrary is known and numeric solutions describing the tachyon dynamics were obtained [49]. This effective field theory description does capture the nonlocality of the SFT. Linearizing the latter lagrangian around the true vacuum one gets a model which is of main concern in the present paper. The SFT inspired forms of function  $\mathcal{F}(\square_g)$ , which have the nonlocal operator  $\exp(\alpha\square_g)$ , where  $\alpha$  is a constant, as a key ingredient, have been considered in [8, 14, 15]. Such functions have infinite number of simple roots and maybe one double root.

The paper is organized as follows. In Section 2 we describe the nonlocal SFT inspired cosmological model and its generalization. In Section 3 we calculate the energy-momentum tensor for different special solutions. Using these formulae we build local actions and the corresponding local Einstein equations. In Section 4 we propose the algorithm to find particular solutions of the nonlocal Einstein equations, solving only local ones, and prove the self-consistence of it. Any solution for the obtained system of differential equations is a particular solution for the initial nonlocal Einstein equations. In Section 5 we summarize the obtained results and propose directions for further investigations.

## 2 Model setup

The four-dimensional action with a quadratic potential, motivated by the string field theory, has been studied in [7, 8, 14, 15, 17]. Such a model appears as a linearization of the SFT inspired model in the neighborhood of an extremum of the potential (see [17] for details). For linear models, solving the nonlocal equations using the technique, proposed in [14], is completely equivalent to solving the equations using the diffusion-like partial differential equations [15]. In [15] it has been shown that to fix the initial data for the partial differential equations one can use the initial data of the local fields. By linearising a nonlinear model about a particular field value, one is able to specify initial data for nonlinear models, which he then evolves into the full nonlinear regime using the diffusion-like equation [15].

In this paper we study nonlocal cosmological models with a quadratic potential, in other words, a linear nonlocal model, which can be described by the following action:

$$S = \int d^4x \sqrt{-g} \alpha' \left( \frac{R}{16\pi G_N} + \frac{1}{2g_o^2} \phi \mathcal{F}(\square_g) \phi - \Lambda \right), \quad (2)$$

where  $G_N$  is the Newtonian constant:  $8\pi G_N = 1/M_P^2$ , where  $M_P$  is the Planck mass,  $\alpha'$  is the string length squared (we do not assume  $\alpha' = 1/M_P^2$ ),  $g_o$  is the string coupling constant. We use the signature  $(-, +, +, +)$ ,  $g_{\mu\nu}$  is the metric tensor,  $R$  is the scalar curvature,  $\Lambda$  is the cosmological constant.

The function  $\mathcal{F}$  is assumed to be an analytic function, therefore, one can represent it by the convergent series expansion:

$$\mathcal{F} = \sum_{n=0}^{\infty} f_n \square_g^n. \quad (3)$$

The function  $\mathcal{F}$  may have infinitely many roots manifestly producing thereby the nonlocality [13, 17].

This model has been studied in [7, 17] with an additional condition that all roots of the function  $\mathcal{F}$  are simple. At the same time the obtained formulae for the nonlocal energy-momentum tensor (formulae (4.1) in [7]) are valid in the case of multiple roots as well and we use them in this paper.

In [8, 14] the special class of functions  $\mathcal{F}(\square_g)$ :

$$\mathcal{F}_{sft}(\square_g) = -\xi^2 \square_g + 1 - c e^{-2\square_g}, \quad (4)$$

where  $\xi$  is a real parameter and  $c$  is a positive constant has been considered. The action with  $\mathcal{F}_{sft}(\square_g)$  is interesting in context of the SFT inspired models. In [15] the model has been generalized and a linear term has been added to the action.

The function  $\mathcal{F}_{sft}(\square_g)$  has a double root if and only if

$$c = \frac{\xi^2}{2e} e^{2/\xi^2}. \quad (5)$$

The double root  $\tilde{J}_0$  is as follows

$$\tilde{J}_0 = \frac{1}{\xi^2} - \frac{1}{2}. \quad (6)$$

At any  $\xi$  and  $c$ , which satisfy (5), the function  $\mathcal{F}_{sft}(J)$  has one and only one double root  $\tilde{J}_0$  and  $\mathcal{F}_{sft}''(\tilde{J}_0) \neq 0$ .

In this paper we consider in detail the case of an arbitrary analytic function  $\mathcal{F}$  with both double and simple roots.

To clarify the interest to consider the case of double roots let us study a trivial example with

$$\mathcal{F}(\square_g) = (\square_g - J_1)(\square_g - J_2). \quad (7)$$

In the Minkowski space-time for  $\phi$ , depending only on time, we obtain the following equation of motion

$$(\partial_t^2 - J_1)(\partial_t^2 - J_2)\phi(t) = 0. \quad (8)$$

This fourth-order differential equation can be written in the form of system of two second order equations:

$$(\partial_t^2 - J_1)\xi(t) = 0, \quad (\partial_t^2 - J_2)\phi(t) = \xi(t). \quad (9)$$

The first equation has the general solution

$$\xi(t) = C_1 e^{\sqrt{J_1}t} + C_2 e^{-\sqrt{J_1}t}, \quad (10)$$

where  $C_1$  and  $C_2$  are constants. So, we get the following second order equation for  $\phi$

$$(\partial_t^2 - J_2)\phi(t) = C_1 e^{\sqrt{J_1}t} + C_2 e^{-\sqrt{J_1}t}. \quad (11)$$

In the non-resonance case (two simple roots  $J_1$  and  $J_2$ ) we get the following general solution

$$\phi(t) = \tilde{C}_1 e^{\sqrt{J_1}t} + \tilde{C}_2 e^{-\sqrt{J_1}t} + \tilde{C}_3 e^{\sqrt{J_2}t} + \tilde{C}_4 e^{-\sqrt{J_2}t}, \quad (12)$$

whereas in the resonance case (one double root  $J_2 = J_1$ ) the general solution is

$$\phi(t) = \tilde{C}_1 e^{\sqrt{J_1}t} + \tilde{C}_2 e^{-\sqrt{J_1}t} + \tilde{C}_3 t e^{\sqrt{J_1}t} + \tilde{C}_4 t e^{-\sqrt{J_1}t}, \quad (13)$$

where  $\tilde{C}_k$  are arbitrary constants. This trivial example shows that behaviors of solutions in the cases of one double root and two simple roots are essentially different and one can not approximate double roots by two simple roots, which are at a very small distance. Resonance phenomenons are important and actively studied in various domains of physics.

### 3 Energy–momentum tensor

#### 3.1 The Einstein equations and energy–momentum tensor

From action (2) we obtain the following equations

$$G_{\mu\nu} = \frac{8\pi G_N}{g_o^2} T_{\mu\nu} - 8\pi G_N \Lambda g_{\mu\nu}, \quad (14)$$

$$\mathcal{F}(\square_g)\phi = 0, \quad (15)$$

where  $G_{\mu\nu}$  is the Einstein tensor,  $T_{\mu\nu}$  is the energy–momentum (stress) tensor [7, 17]:

$$\begin{aligned} T_{\mu\nu} = & \frac{-2g_o^2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( \partial_\mu \square_g^l \phi \partial_\nu \square_g^{n-1-l} \phi + \partial_\nu \square_g^l \phi \partial_\mu \square_g^{n-1-l} \phi - \right. \\ & \left. - g_{\mu\nu} (g^{\rho\sigma} \partial_\rho \square_g^l \phi \partial_\sigma \square_g^{n-1-l} \phi + \square_g^l \phi \square_g^{n-l} \phi) \right), \end{aligned} \quad (16)$$

$$\square_g \equiv \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu. \quad (17)$$

It is easy to check that the Bianchi identity is satisfied on-shell and in a simple case  $\mathcal{F} = f_1 \square_g + f_0$  the usual energy–momentum tensor for the massive scalar field is reproduced. Note that equation (15) is an independent equation consistent with system (14) due to the Bianchi identity.

In an arbitrary metric the energy–momentum tensor (16) can be presented in the following form:

$$T_{\mu\nu} = E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu} (g^{\rho\sigma} E_{\rho\sigma} + V), \quad (18)$$

where

$$E_{\mu\nu} \equiv \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\mu \square_g^l \phi \partial_\nu \square_g^{n-1-l} \phi, \quad (19)$$

$$V \equiv \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \square_g^l \phi \square_g^{n-l} \phi. \quad (20)$$

### 3.2 Energy–momentum tensor for special solutions

Classical solutions to system (14)–(15) were studied and analyzed in [7, 8, 14, 17]. The main idea of finding the solutions to the equations of motion is to start with equation (15) and to solve it, assuming the function  $\phi$  is an eigenfunction of the d'Alembertian operator  $\square_g$ . If  $\square_g \phi = J\phi$ , then such a function  $\phi$  is a solution to (15) if and only if

$$\mathcal{F}(J) = 0. \quad (21)$$

The latter condition is known as the *characteristic* equation. Note that values of roots of  $\mathcal{F}(J)$  do not depend on the metric.

Let us denote simple roots of  $\mathcal{F}$  as  $J_i$  and double roots of  $\mathcal{F}$  as  $\tilde{J}_k$ . A particular solution of equation (15) we seek in the following form

$$\phi_0 = \sum_{i=1}^{N_1} \phi_i + \sum_{k=1}^{N_2} \tilde{\phi}_k, \quad (22)$$

where

$$(\square_g - J_i)\phi_i = 0, \quad (\square_g - \tilde{J}_k)^2 \tilde{\phi}_k = 0. \quad (23)$$

Without loss of generality we assume that for any  $i_1$  and  $i_2 \neq i_1$  conditions  $J_{i_1} \neq J_{i_2}$  and  $\tilde{J}_{i_1} \neq \tilde{J}_{i_2}$  are satisfied. Indeed, if, for example, sum (22) includes two summands  $\phi_{i_1}$  and  $\phi_{i_2}$ , which correspond to one and the same  $J_i$ , then we can consider them as one summand  $\phi_i \equiv \phi_{i_1} + \phi_{i_2}$ , which corresponds to  $J_i$ .

In previous papers [7, 8, 14, 15, 17] only the case of simple roots has been studied. In our paper we generalize this analysis on double roots. Our first goal is to calculate the energy–momentum tensor for  $\phi_0$ . To obtain the general formula we begin from a few particular cases. Hereafter we denote the energy–momentum tensor for the function  $\phi(t)$  as  $T_{\mu\nu}(\phi)$ .

### 3.3 Simple roots

If we have one simple root  $\phi_1$  such that  $\square_g \phi_1 = J_1 \phi_1$ , then

$$E_{\mu\nu}(\phi_1) = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} J_1^{n-1} \partial_\mu \phi_1 \partial_\nu \phi_1 = \frac{\mathcal{F}'(J_1)}{2} \partial_\mu \phi_1 \partial_\nu \phi_1. \quad (24)$$

$$V(\phi_1) = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} J_1^n \phi_1^2 = \frac{J_1}{2} \sum_{n=1}^{\infty} f_n n J_1^{n-1} \phi_1^2 = \frac{J_1 \mathcal{F}'(J_1)}{2} \phi_1^2, \quad (25)$$

where  $\mathcal{F}' \equiv \frac{d\mathcal{F}}{dJ}$ .

In the case of two simple roots  $\phi_1$  and  $\phi_2$  we have

$$E_{\mu\nu}(\phi_1 + \phi_2) = E_{\mu\nu}(\phi_1) + E_{\mu\nu}(\phi_2) + E_{\mu\nu}^{cr}(\phi_1, \phi_2), \quad (26)$$

where the cross term

$$E_{\mu\nu}^{cr}(\phi_1, \phi_2) = A_1 \partial_\mu \phi_1 \partial_\nu \phi_2 + A_2 \partial_\mu \phi_2 \partial_\nu \phi_1. \quad (27)$$

It is easy to calculate that

$$A_1 = \frac{1}{2} \sum_{n=1}^{\infty} f_n J_1^{n-1} \sum_{l=0}^{n-1} \left( \frac{J_2}{J_1} \right)^l = \frac{\mathcal{F}(J_1) - \mathcal{F}(J_2)}{2(J_1 - J_2)} = 0, \quad (28)$$

and

$$A_2 = 0. \quad (29)$$

So, the cross term  $E_{\mu\nu}^{cr}(\phi_1, \phi_2) = 0$  and

$$E_{\mu\nu}(\phi_1 + \phi_2) = E_{\mu\nu}(\phi_1) + E_{\mu\nu}(\phi_2). \quad (30)$$

Similar calculations show

$$V(\phi_1 + \phi_2) = V(\phi_1) + V(\phi_2). \quad (31)$$

In the case of  $N$  simple roots the following formula has been obtained [17] (see also [14]):

$$T_{\mu\nu} = \sum_{k=1}^N \mathcal{F}'(J_k) \left( \partial_\mu \phi_k \partial_\nu \phi_k - \frac{1}{2} g_{\mu\nu} (g^{\rho\sigma} \partial_\rho \phi_k \partial_\sigma \phi_k + J_k \phi_k^2) \right). \quad (32)$$

Note that the last formula is exactly the energy-momentum tensor of many free massive scalar fields. If  $\mathcal{F}(J)$  has simple real roots, then positive and negative values of  $\mathcal{F}'(J_i)$  alternate, so we can obtain phantom fields.

### 3.4 One double root

Let us consider the case, when all roots of  $F(J)$ , but one, are simple and the last root is a double root. As we mentioned above this case is interesting in context of the SFT inspired models.

Let  $\tilde{J}_1$  is a double root. The fourth order differential equation

$$(\square_g - \tilde{J}_1)(\square_g - \tilde{J}_1)\tilde{\phi}_1 = 0 \quad (33)$$

is equivalent to the following system of equations:

$$(\square_g - \tilde{J}_1)\tilde{\phi}_1 = \varphi_1, \quad (\square_g - \tilde{J}_1)\varphi_1 = 0. \quad (34)$$

It is convenient to write  $\square_g^l \tilde{\phi}_1$  in terms of  $\tilde{\phi}_1$  and  $\varphi_1$ :

$$\square_g^l \tilde{\phi}_1 = \tilde{J}_1^l \tilde{\phi}_1 + l \tilde{J}_1^{l-1} \varphi_1. \quad (35)$$

Using (35) we obtain

$$E_{\mu\nu}(\tilde{\phi}_1) = B_1 \partial_\mu \tilde{\phi}_1 \partial_\nu \tilde{\phi}_1 + B_2 \partial_\mu \tilde{\phi}_1 \partial_\nu \varphi_1 + B_3 \partial_\nu \tilde{\phi}_1 \partial_\mu \varphi_1 + B_4 \partial_\mu \varphi_1 \partial_\nu \varphi_1, \quad (36)$$

where

$$B_1 = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \tilde{J}_1^{n-1} = \frac{1}{2} \sum_{n=1}^{\infty} f_n n \tilde{J}_1^{n-1} = \frac{\mathcal{F}'(\tilde{J}_1)}{2} = 0. \quad (37)$$

$$B_2 = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} (n-l-1) \tilde{J}_1^{n-2} = \frac{1}{4} \sum_{n=1}^{\infty} f_n n(n-1) \tilde{J}_1^{n-2} = \frac{\mathcal{F}''(\tilde{J}_1)}{4}. \quad (38)$$

$$B_3 = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} l \tilde{J}_1^{n-2} = \frac{\mathcal{F}''(\tilde{J}_1)}{4} = B_2. \quad (39)$$

$$B_4 = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} (n-l-1) l \tilde{J}_1^{n-3} = \frac{1}{12} \sum_{n=1}^{\infty} n(n-1)(n-2) f_n \tilde{J}_1^{n-3} = \frac{\mathcal{F}'''(\tilde{J}_1)}{12}. \quad (40)$$

We have used the well-known formulae:

$$\sum_{l=0}^{n-1} l = \frac{n(n-1)}{2} \quad \text{and} \quad \sum_{l=0}^{n-1} l^2 = \frac{n(n-1)(2n-1)}{6}. \quad (41)$$

Similar calculations give

$$V(\tilde{\phi}_1) = C_1 \tilde{\phi}_1^2 + C_2 \tilde{\phi}_1 \varphi_1 + C_3 \varphi_1^2, \quad (42)$$

where

$$C_1 = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \tilde{J}_1^n = \frac{\tilde{J}_1}{2} \sum_{n=1}^{\infty} f_n n \tilde{J}_1^{n-1} = \frac{\tilde{J}_1 \mathcal{F}'(\tilde{J}_1)}{2} = 0, \quad (43)$$

$$C_2 = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} n \tilde{J}_1^{n-1} = \frac{\tilde{J}_1 \mathcal{F}''(\tilde{J}_1)}{2} + \frac{\mathcal{F}'(\tilde{J}_1)}{2} = \frac{\tilde{J}_1 \mathcal{F}''(\tilde{J}_1)}{2}, \quad (44)$$

$$C_3 = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} l(n-l) \tilde{J}_1^{n-2} = \frac{\tilde{J}_1 \mathcal{F}'''(\tilde{J}_1)}{12} + \frac{\mathcal{F}''(\tilde{J}_1)}{4}. \quad (45)$$

Thus, for one double root we obtain the following result:

$$E_{\mu\nu}(\tilde{\phi}_1) = \frac{\mathcal{F}''(\tilde{J}_1)}{4} (\partial_\mu \tilde{\phi}_1 \partial_\nu \varphi_1 + \partial_\nu \tilde{\phi}_1 \partial_\mu \varphi_1) + \frac{\mathcal{F}'''(\tilde{J}_1)}{12} \partial_\mu \varphi_1 \partial_\nu \varphi_1, \quad (46)$$

$$V(\tilde{\phi}_1) = \frac{\tilde{J}_1 \mathcal{F}''(\tilde{J}_1)}{2} \tilde{\phi}_1 \varphi_1 + \left( \frac{\tilde{J}_1 \mathcal{F}'''(\tilde{J}_1)}{12} + \frac{\mathcal{F}''(\tilde{J}_1)}{4} \right) \varphi_1^2. \quad (47)$$

For one simple root  $J_2$  (the function  $\phi_2$  satisfies the equation  $\square_g \phi_2 = J_2 \phi_2$ ) and one double root  $\tilde{J}_1$  we obtain:

$$E_{\mu\nu}(\tilde{\phi}_1 + \phi_2) = E_{\mu\nu}(\tilde{\phi}_1) + E_{\mu\nu}(\phi_2) + E_{\mu\nu}^{cr}(\tilde{\phi}_1, \phi_2), \quad (48)$$

where

$$E_{\mu\nu}^{cr}(\tilde{\phi}_1, \phi_2) = B_5 \partial_\mu \tilde{\phi}_1 \partial_\nu \phi_2 + B_6 \partial_\nu \tilde{\phi}_1 \partial_\mu \phi_2 + B_7 \partial_\mu \varphi_1 \partial_\nu \phi_2 + B_8 \partial_\nu \varphi_1 \partial_\mu \phi_2. \quad (49)$$

It is easy to calculate:

$$B_5 = \frac{1}{2} \sum_{n=1}^{\infty} f_n J_2^{n-1} \sum_{l=0}^{n-1} \left( \frac{\tilde{J}_1}{J_2} \right)^l = \frac{\mathcal{F}(J_2) - \mathcal{F}(\tilde{J}_1)}{2(J_2 - \tilde{J}_1)} = 0. \quad (50)$$



$$B_6 = 0. \quad (51)$$

To calculate

$$B_7 = \frac{1}{2} \sum_{n=1}^{\infty} f_n J_2^n \sum_{l=0}^{n-1} l \left( \frac{\tilde{J}_1}{J_2} \right)^l \quad (52)$$

we use

$$\sum_{l=0}^{n-1} l y^{l-1} = \frac{d}{dy} \sum_{l=0}^{n-1} y^l = \frac{d}{dy} \left( \frac{1-y^n}{1-y} \right) = \frac{(n-1)y^n - ny^{n-1} + 1}{(1-y)^2} \quad (53)$$

and obtain

$$B_7 = \frac{J_2^2}{2(J_2 - \tilde{J}_1)} \mathcal{F}'(\tilde{J}_1) + \frac{J_2^2}{2(J_2 - \tilde{J}_1)^2} (\mathcal{F}(\tilde{J}_1) + \mathcal{F}(J_2)) = 0. \quad (54)$$

Similar calculations give

$$B_8 = 0. \quad (55)$$

It is easy to obtain that

$$V(\tilde{\phi}_1 + \phi_2) = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \square_g^l(\tilde{\phi}_1 + \phi_2) \square_g^{n-l}(\tilde{\phi}_1 + \phi_2) = V(\tilde{\phi}_1) + V(\phi_2). \quad (56)$$

The calculations are straightforwardly generalized on the case of one double root and an arbitrary number of simple roots. Therefore, we obtain the following formula

$$T_{\mu\nu} \left( \tilde{\phi}_1 + \sum_{k=1}^N \phi_k \right) = T_{\mu\nu}(\tilde{\phi}_1) + T_{\mu\nu} \left( \sum_{k=1}^N \phi_k \right), \quad (57)$$

where

$$T_{\mu\nu}(\tilde{\phi}_1) = E_{\mu\nu}(\tilde{\phi}_1) + E_{\nu\mu}(\tilde{\phi}_1) - g_{\mu\nu} \left( g^{\rho\sigma} E_{\rho\sigma}(\tilde{\phi}_1) + V(\tilde{\phi}_1) \right). \quad (58)$$

So, we conclude that in the case of one double root the energy-momentum tensor can be separated into energy-momentum tensors for different modes of nonlocal scalar field, which correspond to different roots of  $\mathcal{F}$ .

### 3.5 The general formulae

Let us consider the case of two double roots  $\tilde{J}_1$  and  $\tilde{J}_2$ . We can write

$$E_{\mu\nu}(\tilde{\phi}_1 + \tilde{\phi}_2) = E_{\mu\nu}(\tilde{\phi}_1) + E_{\mu\nu}(\tilde{\phi}_2) + E_{\mu\nu}^{cr}(\tilde{\phi}_1, \tilde{\phi}_2), \quad (59)$$

where

$$\begin{aligned} E_{\mu\nu}^{cr}(\tilde{\phi}_1, \tilde{\phi}_2) = & B_{10} \partial_\mu \tilde{\phi}_1 \partial_\nu \tilde{\phi}_2 + B_{11} \partial_\nu \tilde{\phi}_1 \partial_\mu \tilde{\phi}_2 + B_{12} \partial_\mu \tilde{\phi}_1 \partial_\nu \varphi_2 + \\ & + B_{13} \partial_\nu \tilde{\phi}_1 \partial_\mu \varphi_2 + B_{14} \partial_\mu \varphi_1 \partial_\nu \tilde{\phi}_2 + B_{15} \partial_\nu \varphi_1 \partial_\mu \tilde{\phi}_2 + \\ & + B_{16} \partial_\mu \varphi_1 \partial_\nu \varphi_2 + B_{17} \partial_\nu \varphi_1 \partial_\mu \varphi_2. \end{aligned} \quad (60)$$

Using computations, which are similar to computations of  $B_5$  and  $B_7$ , it is easy to see

$$B_{10} = B_{11} = B_{12} = B_{13} = B_{14} = B_{15} = 0. \quad (61)$$

It is suitable to present

$$B_{16} = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} l(n-l-1) \tilde{J}_1^{l-1} \tilde{J}_2^{n-l-2} \quad (62)$$

in the following form:

$$B_{16} = \frac{1}{2} \sum_{n=1}^{\infty} f_n \tilde{J}_2^{n-1} \sum_{l=0}^{n-1} (n-l-1) l \varpi^{l-1}, \quad (63)$$

where  $\varpi \equiv \tilde{J}_1/\tilde{J}_2$ . Using

$$\sum_{l=0}^{n-1} (n-l-1) l \varpi^{l-1} = n \sum_{l=0}^{n-1} l \varpi^{l-1} - \varpi \sum_{l=0}^{n-1} (l-1) l \varpi^{l-2} = n \frac{d}{d\varpi} \left( \sum_{l=0}^{n-1} \varpi^l \right) - \varpi \frac{d^2}{d\varpi^2} \left( \sum_{l=0}^{n-1} \varpi^l \right)$$

and

$$\sum_{l=0}^{n-1} \varpi^l = \frac{1 - \varpi^n}{1 - \varpi}, \quad (64)$$

we obtain

$$\sum_{l=0}^{n-1} (n-l-1) l \varpi^{l-1} = n \frac{1 + \varpi^{n-1}(2\varpi - 1)}{(\varpi - 1)^2} + 2\varpi \frac{1 - \varpi^n}{(\varpi - 1)^3}. \quad (65)$$

Thus we get

$$B_{16} = \frac{\tilde{J}_2(\mathcal{F}(\tilde{J}_2) - \mathcal{F}(\tilde{J}_1))}{(\tilde{J}_2 - \tilde{J}_1)^3} + \frac{\tilde{J}_2(2\tilde{J}_1 - \tilde{J}_2)\mathcal{F}'(\tilde{J}_1) + \tilde{J}_2^2\mathcal{F}'(\tilde{J}_2)}{2(\tilde{J}_1 - \tilde{J}_2)^2}. \quad (66)$$

So,  $B_{16} = 0$ . The similar calculations prove that  $B_{17} = 0$  and we come to the following result:

$$E_{\mu\nu}(\tilde{\phi}_1 + \tilde{\phi}_2) = E_{\mu\nu}(\tilde{\phi}_1) + E_{\mu\nu}(\tilde{\phi}_2). \quad (67)$$

We also obtain

$$V(\tilde{\phi}_1 + \tilde{\phi}_2) = V(\tilde{\phi}_1) + V(\tilde{\phi}_2). \quad (68)$$

The results, obtained for two summands, can be straightforwardly generalized on an arbitrary number of summands. So, we obtain that for any analytical function  $\mathcal{F}$ , which has simple roots  $J_i$  and double roots  $\tilde{J}_k$ , and any  $\phi_0$  given by (22) the energy-momentum tensor

$$T_{\mu\nu}(\phi_0) = T_{\mu\nu} \left( \sum_{i=1}^{N_1} \phi_i + \sum_{k=1}^{N_2} \tilde{\phi}_k \right) = \sum_{i=1}^{N_1} T_{\mu\nu}(\phi_i) + \sum_{k=1}^{N_2} T_{\mu\nu}(\tilde{\phi}_k). \quad (69)$$

The result has been obtained for an arbitrary metric  $g_{\mu\nu}$ .

Considering the following local action

$$\begin{aligned}
S_{loc} = & \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G_N} - \Lambda - \right. \\
& - \frac{1}{g_o^2} \left( \sum_{i=1}^{N_1} \frac{\mathcal{F}'(J_i)}{2} (g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i + J_i \phi_i^2) - \right. \\
& - \sum_{k=1}^{N_2} \left( g^{\mu\nu} \left( \frac{\mathcal{F}''(\tilde{J}_k)}{2} \partial_\mu \tilde{\phi}_k \partial_\nu \varphi_k + \frac{\mathcal{F}'''(\tilde{J}_k)}{12} \partial_\mu \varphi_k \partial_\nu \varphi_k \right) + \right. \\
& \left. \left. + \frac{\tilde{J}_k \mathcal{F}''(\tilde{J}_k)}{2} \tilde{\phi}_k \varphi_k + \left( \frac{\tilde{J}_k \mathcal{F}'''(\tilde{J}_k)}{12} + \frac{\mathcal{F}''(\tilde{J}_k)}{4} \right) \varphi_k^2 \right) \right) \Bigg), \tag{70}
\end{aligned}$$

we can see that solutions of the system of the Einstein equations and equations in  $\phi_k$ ,  $\tilde{\phi}_k$  and  $\varphi_k$ , obtained from this action, solves the initial system of nonlocal equations (14) and (15). Thus, we obtained that special solutions of nonlocal equations one can find solving system of local (differential) equations.

To clarify physical interpretation we diagonalize the kinetic terms of scalar fields  $\tilde{\phi}_k$  and  $\varphi_k$  in action (70). It is convenient to present (70) as follows:

$$S_{loc} = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G_N} - \Lambda \right) + \sum_{i=1}^{N_1} S_i + \sum_{k=1}^{N_2} \tilde{S}_k, \tag{71}$$

where

$$S_i = - \frac{1}{2g_o^2} \int d^4x \sqrt{-g} \mathcal{F}'(J_i) (g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i + J_i \phi_i^2), \tag{72}$$

$$\begin{aligned}
\tilde{S}_k = & - \frac{1}{g_o^2} \int d^4x \sqrt{-g} \left[ g^{\mu\nu} \left( \frac{\mathcal{F}''(\tilde{J}_k)}{4} (\partial_\mu \tilde{\phi}_k \partial_\nu \varphi_k + \partial_\nu \tilde{\phi}_k \partial_\mu \varphi_k) + \right. \right. \\
& + \left. \frac{\mathcal{F}'''(\tilde{J}_k)}{12} \partial_\mu \varphi_k \partial_\nu \varphi_k \right) + \\
& + \left. \frac{\tilde{J}_k \mathcal{F}''(\tilde{J}_k)}{2} \tilde{\phi}_k \varphi_k + \left( \frac{\tilde{J}_k \mathcal{F}'''(\tilde{J}_k)}{12} + \frac{\mathcal{F}''(\tilde{J}_k)}{4} \right) \varphi_k^2 \right], \tag{73}
\end{aligned}$$

Expressing  $\tilde{\phi}_k$  and  $\varphi_k$  in terms of new fields  $\xi_k$  and  $\chi_k$ :

$$\tilde{\phi}_k = \frac{1}{2\mathcal{F}''(\tilde{J}_k)} \left( (\mathcal{F}''(\tilde{J}_k) - \frac{1}{3}\mathcal{F}'''(\tilde{J}_k))\xi_k - (\mathcal{F}''(\tilde{J}_k) + \frac{1}{3}\mathcal{F}'''(\tilde{J}_k))\chi_k \right), \tag{74}$$

$$\varphi_k = \xi_k + \chi_k, \tag{75}$$

we obtain the corresponding  $\tilde{S}_k$  in the following form:

$$\begin{aligned}\tilde{S}_k = & -\frac{1}{g_o^2} \int d^4x \sqrt{-g} \left[ g^{\mu\nu} \frac{\mathcal{F}''(\tilde{J}_k)}{4} (\partial_\mu \xi_k \partial_\nu \xi_k - \partial_\nu \chi_k \partial_\mu \chi_k) + \right. \\ & + \frac{\tilde{J}_k}{4} \left( \left( \mathcal{F}''(\tilde{J}_k) - \frac{1}{3} \mathcal{F}'''(\tilde{J}_k) \right) \xi_k - \left( \mathcal{F}''(\tilde{J}_k) + \frac{1}{3} \mathcal{F}'''(\tilde{J}_k) \right) \chi_k \right) (\xi_k + \chi_k) + \\ & \left. + \left( \frac{\tilde{J}_k \mathcal{F}'''(\tilde{J}_k)}{12} + \frac{\mathcal{F}''(\tilde{J}_k)}{4} \right) (\xi_k + \chi_k)^2 \right].\end{aligned}$$

It is easy to see that each  $\tilde{S}_k$  includes one phantom scalar field and one standard scalar field. So, in the case of one double root we obtain a quintom model. In the Minkowski space appearance of phantom fields in models, when  $\mathcal{F}(\square)$  has a double root, has been obtained in [46].

## 4 The algorithm of localization

The obtained formulae allow us to seek particular solutions for nonlocal gravitational models with quadratic potentials, which described by action (2), in the following way:

- Find roots of the function  $\mathcal{F}(J)$  and calculate orders of them.
- Select an finite number of simple and double roots.
- Construct the corresponding local action by formula (70).
- Obtain a system of the Einstein equations and equations of motion. The obtained system is a finite order system of differential equations, in other words we get a local system.
- Seek solutions of the obtained local system.

**Remark 1.** If  $\mathcal{F}(J)$  has an infinity number of roots then one nonlocal model corresponds to infinity number of different local models. In this case the initial nonlocal action (2) generates infinity number of local actions (70).

**Remark 2.** We should prove that our algorithm is self-consistent. To construct local action (70) we assume that equations (23) are satisfied. Therefore, our algorithm is correct only if these equations can be obtained from the local action (70). The straightforward calculations show that

$$\frac{\delta S_{loc}}{\delta \phi_i} = 0 \quad \Leftrightarrow \quad \square_g \phi_i = J_i \phi_i, \quad (76)$$

$$\frac{\delta S_{loc}}{\delta \tilde{\phi}_k} = 0 \quad \Leftrightarrow \quad \square_g \varphi_k = \tilde{J}_k \varphi_k. \quad (77)$$

Using (77) we obtain

$$\frac{\delta S_{loc}}{\delta \varphi_k} = 0 \quad \Leftrightarrow \quad \square_g \tilde{\phi}_k = \tilde{J}_k \tilde{\phi}_k + \varphi_k. \quad (78)$$

So, our way of localization is self-consistent in the case of  $\mathcal{F}(J)$  with simple and double roots. The self-consistence of similar approach for  $\mathcal{F}(J)$  with only simple roots has been proven in [14, 17].

In spite of the above-mention equations we obtain from  $S_{loc}$  the Einstein equations:

$$G_{\mu\nu} = \frac{8\pi G_N}{g_o^2} T_{\mu\nu}(\phi_0) - 8\pi G_N \Lambda g_{\mu\nu}, \quad (79)$$

where  $\phi_0$  is given by (22) and  $T_{\mu\nu}(\phi_0)$  can be calculated by (69).

So, we obtained such systems of differential equations that any solutions of these systems are particular solutions of the initial nonlocal equations (14) and (15).

## 5 Conclusion

The main result of this paper is the explicit proof that nonlocal cosmological model can be localized not only in the case, when  $\mathcal{F}(\square_g)$  has only simple roots. We have found the way to find particular solutions of the nonlocal Einstein equations in the case when an analytic function  $\mathcal{F}(\square_g)$  has both simple and double roots. We prove that the same functions solve the initial nonlocal Einstein equations and the obtained local Einstein equations. We have found the corresponding local actions and proved the self-consistence of our approach. The result has been obtained for an arbitrary metric, so it can be used not only to find solutions in the Friedmann–Robertson–Walker metric, but also to find other interesting solutions, for example, black hole solutions. In the case of simple roots some exact solutions in the Friedmann–Robertson–Walker metric have been found in [14] (the stability of these solutions is considered in [52]).

Looking a step further it is interesting to consider nonlocal models with an arbitrary analytic  $\mathcal{F}(\square_g)$ , without any restrictions on order of roots. The consideration of simple roots in papers [14, 17] and double roots in this paper allows us to make the conjecture that the existence of local actions, which correspond to a nonlocal action, does not depend on order of  $\mathcal{F}(\square_g)$  roots and the method of finding particular solutions of the nonlocal Einstein equations can be generalized on a nonlocal action with an arbitrary analytic  $\mathcal{F}(\square_g)$ .

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